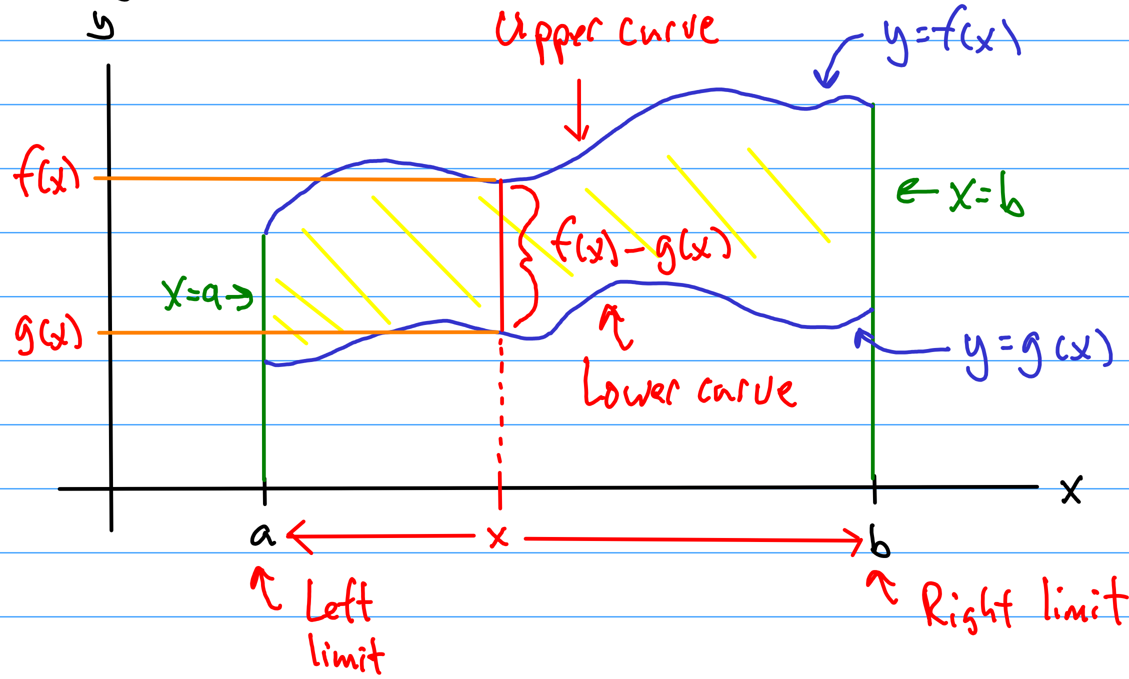
Area Between Curves

- Suppose f and g are continuous functions on $[a,b]$ with antiderivatives F and G respectively. Suppose $f(x) \geq g(x)$ for all x in $[a,b]$.

- 

- The shaded region is bounded by $y=f(x)$, $y=g(x)$, $x=a$, and $x=b$.

- Area of shaded region = A

$$= \underbrace{\int_a^b f(x)\,dx}_{\substack{\text{Area under} \\ y=f(x) \text{ between} \\ x=a \text{ and } x=b}} - \underbrace{\int_a^b g(x)\,dx}_{\substack{\text{Area under} \\ y=g(x) \text{ between} \\ x=a \text{ and } x=b}}$$

$$= \int_{\underset{\text{Left limit}}{a}}^{b \leftarrow \text{Right limit}} \underbrace{f(x) - g(x)}_{\text{Length of line segment}} \, dx$$

x is changing

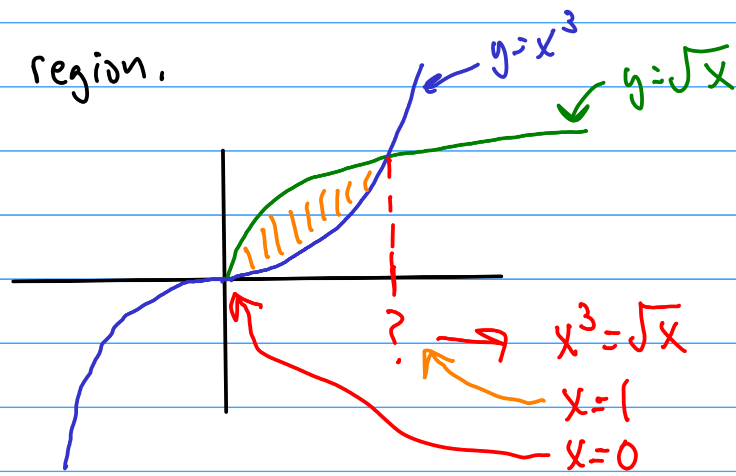
- The area of the region is the sum of the lengths of all of the line segments. To find that sum we integrate.

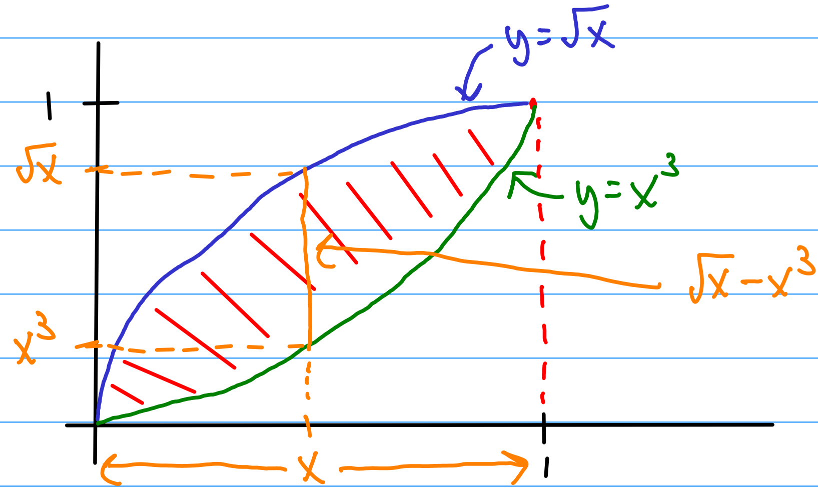
- $$A = \int_{\text{Left limit}}^{\text{Right limit}} \text{Upper curve} - \text{Lower curve} \; dx$$

- **Example:** Find the area of the region bounded by $y = \sqrt{x}$ and $y = x^3$.

- 1) Sketch the region.

Rough



- 

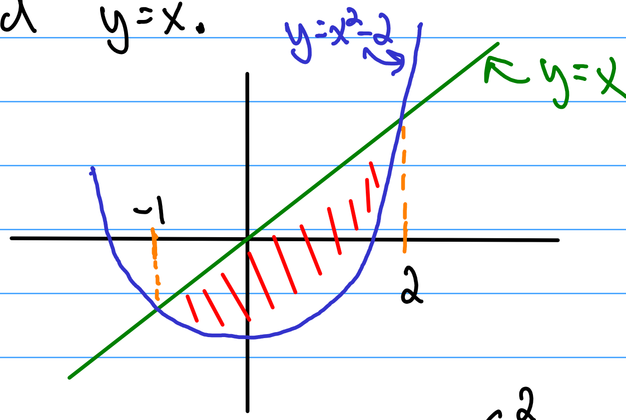
2) Find the area

$$A = \int_0^1 \sqrt{x} - x^3 \, dx$$

$$= \int_0^1 x^{1/2} - x^3 \, dx = \frac{2}{3}x^{3/2} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Example: Find the area of the region bounded by $y = x^2 - 2$ and $y = x$.



$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, \; x = 2$

$$A = \int_{-1}^{2} x - (x^2 - 2) \, dx = \int_{-1}^{2} x - x^2 + 2 \, dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} + 2x \Big|_{-1}^{2} = 2 - \frac{8}{3} + 4 - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \frac{9}{2}$$