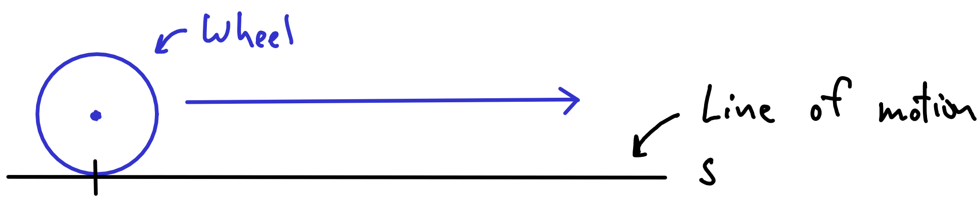
Velocity

(1) **Definition:** A moving object is in linear motion if it is moving along a straight line.

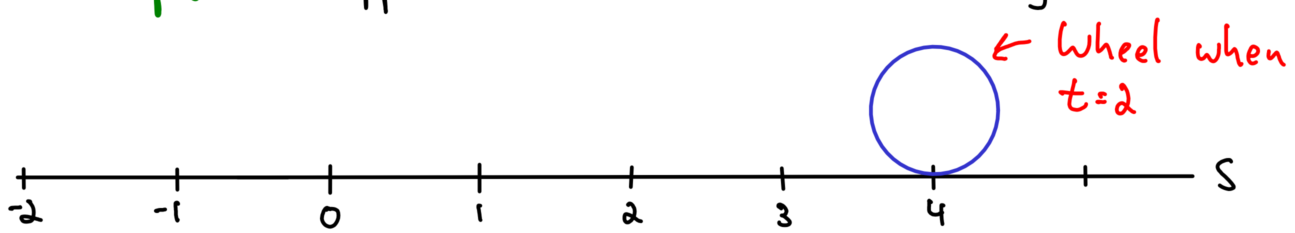
(2) Suppose a wheel is in linear motion.



We will use the variable s to denote the position of the wheel on the line and t to denote time. So s is a function of t.

$$s = f(t)$$

(3) **Example:** Suppose $s = t^2$. When $t = 2$, then $s = 4$.



It is important to note the it is possible for s to be negative since s is the POSITION of the wheel on the line of motion. It is also possible for t to be negative. In addition, we will not associate units of length with s and units of time with t unless those units are part of the problem being discussed.

(4) An object in linear motion changes its position as time changes (unless it is not moving). The rate at which the position of the object is changing as time changes is called the velocity of the object.

(5) **Definition:** The average velocity $\bar{v}$ ("v bar") of an object in linear motion is:

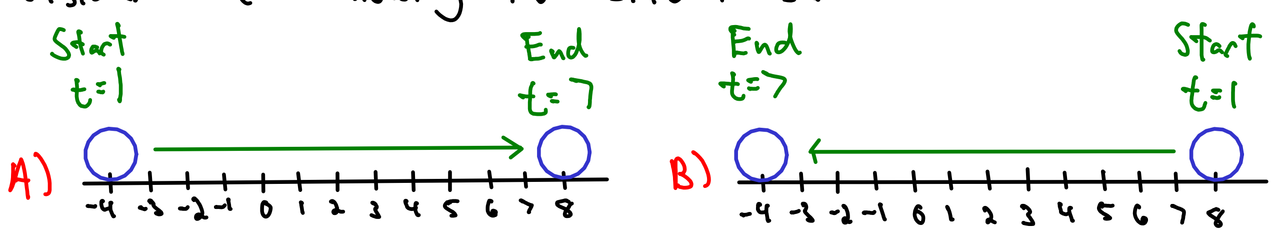
$$\bar{v} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} = \frac{\text{End position} - \text{Start position}}{\text{End time} - \text{Start time}}$$

(6) Average velocity is a rate of change. A rate of change is a ratio of two changing quantities. The slope of a line is also a rate of change.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(7) It is important to realize that velocity is not speed. Speed is always nonnegative, but it is possible for velocity to be negative. Speed is the absolute value of velocity.

(8) What does it mean for an object to have negative velocity? Consider the following two situations.



In A, $\Delta s = 8-(-4) = 12$ and $\Delta t = 7-1=6$, which means that $\bar{v} = 12/6 = 2$. In B, $\Delta s = -4-8=-12$ and $\Delta t = 7-1=6$, which means that $\bar{v} = -12/6 = -2$.

In A, the position of the wheel has increased from -4 to 8 giving a positive average velocity. In B, the position of the wheel has decreased from 8 to -4 giving a negative average velocity. It is clear that average velocity is negative when the position of the object decreases.

Note that the average speed of the wheel in both A and B is 2.

(9) **Example:** Suppose an object is in linear motion and $s=t^2$. Find $\bar{v}$ from $t=0$ to $t=3$.

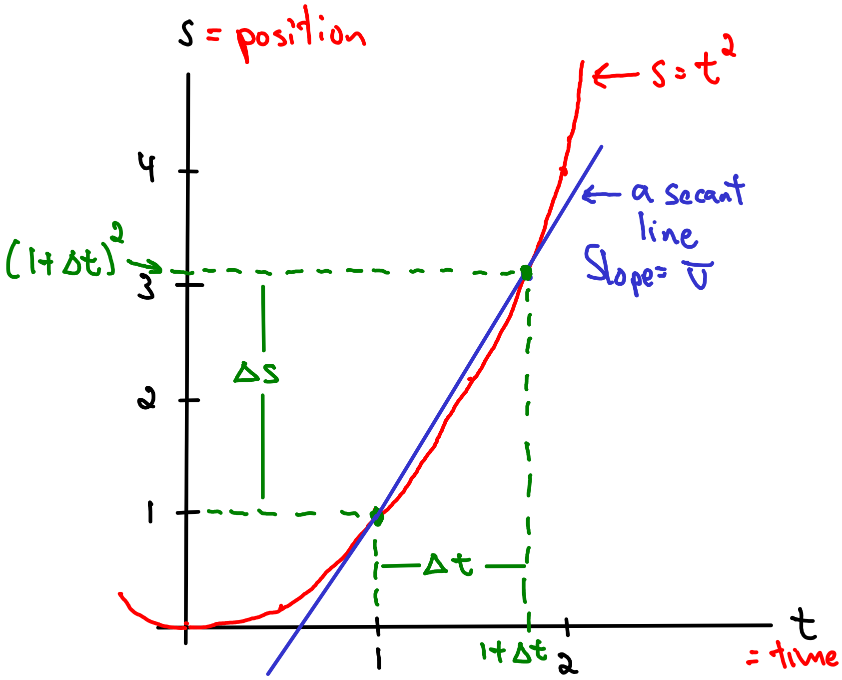
$$s = f(t) = t^2$$
$$\Delta t = 3-0=3$$
$$\Delta s = f(3)-f(0) = 3^2-0^2=9$$
$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{9}{3} = 3$$

What about units? If t is measured in seconds and s is measured in meters, then $\Delta t = 3$ seconds and $\Delta s = 9$ meters so $\bar{v} = 3$ meters/second.

(10) The speedometer of a vehicle does not measure average speed, it measures instantaneous speed. Instantaneous speed is the speed of an object at a single moment in time. Instantaneous speed is the absolute value of the instantaneous velocity. Instantaneous velocity is the velocity of an object at a single moment of time, while average velocity requires an interval of time. Our goal is to compute instantaneous velocity.

(11) Suppose an object is in linear motion and $s = f(t) = t^2$. What is the instantaneous velocity v of the object when $t = 1$?

To answer this question we will examine the graph of $s = t^2$.



First, we will compute $\bar{v}$ from $t=1$ to $t = 1+\Delta t$.

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{(1+\Delta t)^2 - 1}{\Delta t}$$

Notice that $\bar{v}$ is the slope of a secant line of $s = t^2$.

If Δt is very small, then we expect $\bar{v}$ to be close to v.

You don't expect the speed of your car to change by a large amount after one nanosecond (a small Δt).

In the language of calculus, if Δt approaches 0, then $\bar{v}$ approaches v. In other words:

$$v = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{(1+\Delta t)^2 - 1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1 + 2\Delta t + \Delta t^2 - 1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2\Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta t\,(2 + \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 2 + \Delta t$$

$$= 2 + 0$$

$$= 2$$

We have shown that $v = 2$ when $t = 1$.

(12) Notice that since $\bar{v}$ is the slope of a secant line, then v is the slope of a tangent line.

$$v = m_{tan}$$

(13) **Definition:** If $s = f(t)$ is the position at time t of an object in linear motion, then the (instantaneous) velocity v of the object is:

$$v = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

(14) **Example:** ① Suppose an object is in linear motion and $s = 1 - t^2$. Compute v.

$$s = f(t) = 1 - t^2$$

$$v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overbrace{\left[1 - (t+\Delta t)^2\right]}^{f(t+\Delta t)} - \overbrace{\left[1 - t^2\right]}^{f(t)}}{\Delta t}$$

We cannot substitute 0 for Δt yet. We must eliminate

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \lim_{\Delta t \to 0} \frac{1 - \overbrace{(t^2 + 2t\Delta t + \Delta t^2)}^{(t+\Delta t)^2} - 1 + t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1 - t^2 - 2t\Delta t - \Delta t^2 - 1 + t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-2t\Delta t - \Delta t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta t(-2t - \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \underbrace{-2t - \Delta t} = -2t - 0 = -2t$$

No Δt in the denominator

$$v = -2t$$

(15) ② What is v when $t = 3$?

$$v = -2(3) = -6$$

Notice that v is negative when $t=3$. This means the position of the object is decreasing when $t=3$.

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